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Local Hamiltonian structures of multicomponent KdV equations

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Abstract. We discuss the behaviour of Hamiltonian structures for a generalised AKNS hierarchy under componentwise reductions to equations of KdV and MKdV type, identifying those cases where a new local structure arises.

Amongst the many attributes of integrable PDE in one time and one space dimension, such as their construction from infinite-dimensional algebras and their solvability via the inverse scattering transform, is the fact that they possess hierarchies of Hamiltonian structures [1]. These hierarchies are infinite and all but a finite number of their members are non-local, i.e. involve the formal operator ∂_x^{-1}, although the PDE themselves need not be non-local. The nth-order symmetry of such a PDE is thus expressed in all the ways:

$$u_{,t_n} = \omega_m(\nabla_u \mathbb{H}_{n-m})$$

for $n, m \in \mathbb{Z}$ and $n > m$, where $\{\omega_n\}$ is the hierarchy of Hamiltonian structures and $\{\mathbb{H}_n\}$ the hierarchy of Hamiltonian functionals. It is generally the local Hamiltonian structures which are taken to be of interest.

In this paper we discuss the Hamiltonian structures of certain reductions of classes of multicomponent generalisations of the AKNS hierarchy: those associated with Hermitian symmetric spaces. These are discussed in more detail elsewhere [2, 3]. We shall see that the existence of a local Hamiltonian structure is by no means characteristic of these reduced generalised equations. The only locally Hamiltonian equations turn out to be the square matrix KdV equations.

Firstly, in order to motivate the proceedings, we discuss the simple p, q system associated with the 2×2 AKNS problem [4] and its KdV and MKdV reductions.

The AKNS linear problem

$$\left(\partial_x + \frac{1}{2}\begin{pmatrix} a\zeta & 0 \\ 0 & -a\zeta \end{pmatrix} + \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}\right)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 \tag{1}$$

gives rise to non-linear evolution equations, of which the two simplest non-trivial ones are

$$aq_{,t_2} = q_{,xx} - 2q^2 p \qquad -ap_{,t_2} = p_{,xx} - 2p^2 q \tag{2}$$

$$a^2 q_{,t_3} = q_{,xxx} - 6pqq_{,x} \qquad a^2 p_{,t_3} = p_{,xxx} - 6qpp_{,x}. \tag{3}$$

This hierarchy of flows is generated from the trivial one, $q_{,t_1} = q_{,x}$ and $p_{,t_1} = p_{,x}$, by a non-local recursion operator which is easily derived from the hierarchy of linear problems. In fact,

$$\begin{pmatrix} q \\ p \end{pmatrix}_{,t_{n+1}} = a^{-1}\begin{pmatrix} \partial - 2q\partial^{-1}p & -2q\partial^{-1}q \\ 2p\partial^{-1}p & -\partial + 2p\partial^{-1}q \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}_{,t_n} \tag{4}$$

where $\partial = \partial_x = \partial_{t_1}$. We call this recursion operator $\Re$. Each equation has a single local Hamiltonian structure:

$$\begin{pmatrix} q \\ p \end{pmatrix}_{,t_n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta/\delta q \\ \delta/\delta p \end{pmatrix} \mathbb{H}_n = \omega_0(\mathbb{H}_n). \tag{5}$$

Hence by (4) we obtain an infinite hierarchy of such structures [1] whereby any flow may be derived from any Hamiltonian:

$$\begin{array}{cccccccc} & \mathbb{H}_1 & \mathbb{H}_2 & \mathbb{H}_3 & \mathbb{H}_4 & \cdot & \cdot \\ t_1 & \omega_0 & \Re^{-1}\omega_0 & \Re^{-2}\omega_0 & \Re^{-3}\omega_0 & \cdot & \cdot \\ t_2 & \Re\omega_0 & \omega_0 & \Re^{-1}\omega_0 & \Re^{-2}\omega_0 & \cdot & \cdot \\ t_3 & \Re^2\omega_0 & \Re\omega_0 & \omega_0 & \Re^{-1}\omega_0 & \cdot & \cdot \\ t_4 & \Re^3\omega_0 & \Re^2\omega_0 & \Re\omega_0 & \omega_0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \tag{6}$$

or

$$\begin{pmatrix} q \\ p \end{pmatrix}_{,t_n} = \Re^{n-m}\omega_0(\mathbb{H}_m) = \omega_{n-m}(\mathbb{H}_m) \tag{7}$$

where $n, m > 0$.

The hierarchy associated with (2) and (3) has certain admissible reductions. For the odd-order flows one may take $p = \pm q$:

$$a^2 q_{,t_3} = q_{,xxx} \mp 6q^2 q_{,x} \tag{8}$$

(the modified KdV equation [5]) or one may take $p = -1$:

$$a^2 q_{,t_3} = q_{,xxx} + 6qq_{,x} \tag{9}$$

(the KdV equation [5]). A consequence of either of these reductions is that the odd-numbered Hamiltonians in (6) vanish and so we lose the local Hamiltonian structure of the p, q system.

Both equations (8) and (9), however, have local Hamiltonian structures [6]; indeed, equation (9) has two and they arise from the non-local structures $\Re\omega_0$ and $\Re^{-1}\omega_0$ in the following ways.

For the MKdV reduction, put $q = v + \eta$ and $p = v - \eta$ into the ω_1 structure and note that

$$\mathbb{H}_{2n}[v+\eta, v-\eta] = \mathbb{H}_{2n}[v, v] + \mathrm{O}(\eta^2). \tag{10}$$

(The behaviour of the Hamiltonian functionals under (M)KdV type reductions is considered in appendix 1 for the general multicomponent case.) Consequently as $\eta \to 0$ we retain O(1) terms only:

$$\begin{pmatrix} v \\ v \end{pmatrix}_{,t_3} = \frac{a^{-1}}{2} \begin{pmatrix} \partial - 2v\partial^{-1}v & -2v\partial^{-1}v \\ 2v\partial^{-1}v & -\partial + 2v\partial^{-1}v \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta/\delta v \\ \delta/\delta v \end{pmatrix} \mathbb{H}_2[v, v] \tag{11}$$

and the non-local terms cancel to leave the Hamiltonian structure:

$$v_{,t_3} = \frac{a^{-1}}{2} \partial \frac{\delta}{\delta v} \mathbb{H}_2[v, v]. \tag{12}$$

For the KdV reduction put $p=-1+\eta$. In this case $\mathbb{H}_{2n}[q,-1+\eta]=\mathbb{H}_{2n}[q,-1]+\mathbb{H}'_{2n}[q,\eta]+\mathrm{O}(\eta^2)$, where $\mathbb{H}'_{2n}[q,\eta]$ is $\mathrm{O}(\eta)$ as $\eta\to 0$. Now the functional derivative $\delta/\delta p=\delta/\delta\eta$ acting on this $\mathrm{O}(\eta)$ term gives rise to an $\mathrm{O}(1)$ term:

$$\frac{\delta\mathbb{H}'_2}{\delta\eta}=-\tfrac{1}{2}\partial(\partial+2\partial^{-1}q)\frac{\delta\mathbb{H}_2}{\delta q}. \tag{13}$$

Again, the non-local terms cancel to leave the form

$$q_{,t_3}=-\tfrac{1}{2}a^{-1}(\partial^3+2q\partial+2\partial q)\frac{\delta\mathbb{H}_2}{\delta q}. \tag{14}$$

The other Hamiltonian structure arises from

$$\begin{pmatrix} q\\ p\end{pmatrix}_{,t_3}=\Re^{-1}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\delta/\delta q\\ \delta/\delta p\end{pmatrix}\mathbb{H}_4[p,q] \tag{15}$$

under the same reduction. Multiplying by $\Re$ and taking the $\mathrm{O}(1)$ terms gives

$$a^{-1}\begin{pmatrix}\partial+2q\partial^{-1} & -2q\partial^{-1}q\\ 2\partial^{-1} & -\partial-2\partial^{-1}q\end{pmatrix}\begin{pmatrix}q\\ 0\end{pmatrix}_{,t_3}=\begin{pmatrix}\delta\mathbb{H}'_4/\delta\eta\\ -\delta\mathbb{H}_4/\delta q\end{pmatrix} \tag{16}$$

and so, without involving $\mathbb{H}'_4$ at all, we have the Hamiltonian structure:

$$q_{,t_3}=-\frac{a}{2}\partial\frac{\delta}{\delta q}\mathbb{H}_4[q,-1]. \tag{17}$$

It is not difficult to see, by application of the square of the reduced recursion operator in each case, that there are no other local Hamiltonian structures (assuming, of course, that every such structure for the reduced equation must arise from one for the unreduced equation).

Next we examine this situation for some multicomponent (M)KdV equations.

A generalised AKNS linear problem of the following form is taken:

$$(\partial_x+\zeta A+Q)\Psi=0 \tag{18}$$

where Ψ is an N-component vector and A and Q are $N\times N$ matrices. A is diagonal and traceless:

$$A=\left(\begin{array}{c|c}\dfrac{an}{n+m}I_m & 0\\ \hline 0 & \dfrac{-am}{n+m}I_n\end{array}\right)\qquad n+m=N \tag{19}$$

where I_l is the unit $l\times l$ matrix, so that Ad A is highly degenerate. The action of Ad A defines subspaces $\mathfrak{k}, \mathfrak{m}^+$ and $\mathfrak{m}^-$ of dimensions n^2+m^2-1, nm and mn respectively such that

$$[A,\mathfrak{k}]=0\qquad [A,\mathfrak{m}^\pm]=\pm a\mathfrak{m}^\pm \tag{20}$$

and Q is taken to belong to $\mathfrak{m}^+\oplus\mathfrak{m}^-$ and hence to be of the form

$$Q=Q^++Q^-=\left(\begin{array}{c|c}0 & \boldsymbol{q}\\ \hline \boldsymbol{p} & 0\end{array}\right) \tag{21}$$

where $\boldsymbol{q}$ and $\boldsymbol{p}$ are $m\times n$ and $n\times m$ matrices respectively. For more details see [2, 3].

This AKNS system gives rise to the following lowest-order equations:

$$Q^{\pm}_{,t_1} = Q^{\pm}_{,x} \tag{22}$$

$$\pm a Q^{\pm}_{,t_2} = Q^{\pm}_{,xx} - 2[Q^{\pm}, [Q^{\mp}, Q^{\pm}]] \tag{23}$$

$$a^2 Q^{\pm}_{,t_3} = Q^{\pm}_{,xxx} - 3[Q^{\pm}_{,x}, [Q^{\mp}, Q^{\pm}]]. \tag{24}$$

The description (6) and (7) remains true where the local structure is now [7]

$$Q^{\pm}_{,t_n} = (\mathrm{Ad}\, A)\nabla_{Q^{\pm}}\mathbb{H}_n \tag{25}$$

and the recursion operator is

$$\mathfrak{R} = (\partial - (\mathrm{Ad}\, Q)\partial^{-1}(\mathrm{Ad}\, Q))(\mathrm{Ad}\, A)^{-1} \tag{26}$$

so that

$$a\boldsymbol{q}_{,t_3} = \partial(\nabla_p \mathbb{H}_2) - \boldsymbol{q}\partial^{-1}(\boldsymbol{p}\nabla_p\mathbb{H}_2 - \nabla_q\mathbb{H}_2\boldsymbol{q}) + \partial^{-1}(\boldsymbol{q}\nabla_q\mathbb{H}_2 - \nabla_p\mathbb{H}_2\boldsymbol{p})\boldsymbol{q} \tag{27}$$

$$a\boldsymbol{p}_{,t_3} = \partial(\nabla_q \mathbb{H}_2) - \boldsymbol{p}\partial^{-1}(\boldsymbol{q}\nabla_q\mathbb{H}_2 - \nabla_p\mathbb{H}_2\boldsymbol{p}) + \partial^{-1}(\boldsymbol{p}\nabla_p\mathbb{H}_2 - \nabla_q\mathbb{H}_2\boldsymbol{q})\boldsymbol{p} \tag{28}$$

where ∂^{-1} acts only on the quantities in brackets, or

$$aq_{ij,t_3} = (\delta_{il}\delta_{jk}\partial - q_{im}\partial^{-1}p_{ml}\delta_{jk} - q_{mj}\partial^{-1}p_{km}\delta_{il})\nabla_{p_{kl}}\mathbb{H}_2 + (q_{il}\partial^{-1}q_{kj} + q_{kj}\partial^{-1}q_{il})\nabla_{q_{kl}}\mathbb{H}_2 \tag{29}$$

$$ap_{ij,t_3} = (\delta_{il}\delta_{jk}\partial - p_{im}\partial^{-1}q_{ml}\delta_{kj} - p_{mj}\partial^{-1}q_{km}\delta_{il})\nabla q_{kl}\mathbb{H}_2 + (p_{il}\partial^{-1}p_{kj} + p_{kj}\partial^{-1}p_{il})\nabla_{p_{kl}}\mathbb{H}_2 \tag{30}$$

where ∂^{-1} now acts to the right.

Note that if we define ∂^{-1} to be $\int_{-\infty}^{x} \mathrm{d}x$, assuming all components of $\boldsymbol{p}$ and $\boldsymbol{q}$ and all their x derivatives to be functions vanishing as $x \to -\infty$, then $\mathfrak{R}$ has a formal inverse in the algebra of pseudodifferential operators, also with domain $\mathfrak{m}$:

$$\mathfrak{R}^{-1} = (\mathrm{Ad}\, A)\partial^{-1} \sum_{n=0}^{\infty} ((\mathrm{Ad}\, Q)\partial^{-1})^{2n}. \tag{31}$$

For the MKdV type reduction, $q_{ij} = v_{ij} + \eta_{ij}$ and $p_{ij} = v_{ji} - \eta_{ji}$, we have (appendix 1) $\mathbb{H}_{2n+1} = \mathrm{O}(\eta)$, $\mathbb{H}_{2n} = \mathbb{H}_{2n}[v, v^{\mathrm{T}}] + \mathrm{O}(\eta^2)$ as $\eta \to 0$ and then

$$\begin{aligned} av_{ij,t_3} = {}& \tfrac{1}{2}(\delta_{il}\delta_{jk}\partial - v_{im}\partial^{-1}v_{lm}\delta_{jk} - v_{mj}\partial^{-1}v_{mk}\delta_{il})\nabla_{v_{lk}}\mathbb{H}_2 \\ & + \tfrac{1}{2}(v_{il}\partial^{-1}v_{kj} + v_{kj}\partial^{-1}v_{il})\nabla_{v_{kl}}\mathbb{H}_2. \end{aligned} \tag{32}$$

The other equation of the pair (29) and (30) gives the same equation under the permutation $(ij)(lk)$. So the reduced Hamiltonian structure is

$$\tfrac{1}{2}(\delta_{il}\delta_{jk}\partial - v_{im}\partial^{-1}v_{lm}\delta_{jk} - v_{mj}\partial^{-1}v_{mk}\delta_{il} + v_{ik}\partial^{-1}v_{lj} + v_{lj}\partial^{-1}v_{ik}). \tag{33}$$

By performing contractions over j, k and i, l in the non-local part of (33) one sees that it can vanish only when $\boldsymbol{v}$ is a one-component matrix. Therefore, within this class of generalisations, only the one-component MKdV has a local Hamiltonian structure.

The KdV type reduction is of slightly more interest. In (27) and (28) we put $\boldsymbol{p} = -\boldsymbol{r} + \eta$, where $\boldsymbol{r}$ is a constant matrix. Then

$$\mathbb{H}_{2n}[\boldsymbol{q}, \boldsymbol{p}] = \mathbb{H}_{2n}[\boldsymbol{q}, -\boldsymbol{r}] + \mathbb{H}'_{2n}[\boldsymbol{q}, -\boldsymbol{r}, \eta] + \mathrm{O}(\eta^2)$$

for small η and $\mathbb{H}'_{2n}$ is $\mathrm{O}(\eta)$. Then $\nabla_\eta\mathbb{H}'_{2n}$ is $\mathrm{O}(1)$ and

$$0 = \partial(\nabla_q\mathbb{H}_2) + \boldsymbol{r}\partial^{-1}(\boldsymbol{q}\nabla_q\mathbb{H}_2) + \partial^{-1}(\nabla_q\mathbb{H}_2\boldsymbol{q})\boldsymbol{r} + 2\boldsymbol{r}\partial^{-1}(\nabla_\eta\mathbb{H}'_2)\boldsymbol{r} \tag{34}$$

so that, only in the case where $\boldsymbol{r}$ and $\boldsymbol{q}$ are square matrices, we can write

$$\nabla_\eta\mathbb{H}'_2 = -\tfrac{1}{2}[\boldsymbol{r}^{-1}\partial^2(\nabla_q\mathbb{H}'_2)\boldsymbol{r}^{-1} + \boldsymbol{q}\nabla_q\mathbb{H}_2\boldsymbol{r}^{-1} + \boldsymbol{r}^{-1}\nabla_q\mathbb{H}_2\boldsymbol{q}] \tag{35}$$

and

$$aq_{,t_3} = -\tfrac{1}{2}[r^{-1}\partial^3(\nabla_q \mathbb{H}_2)r^{-1} + \partial(q\nabla_q \mathbb{H}_2)r^{-1} + q\partial(\nabla_q \mathbb{H}_2)r^{-1} + r^{-1}\partial(\nabla_q \mathbb{H}_2 q) + r^{-1}\partial(\nabla_q \mathbb{H}_2)q] + \text{NL}. \tag{36}$$

The non-local part, NL, of the Hamiltonian structure is given by

$$\text{NL} = -\tfrac{1}{2}[q, \partial^{-1}[q, \nabla_q \mathbb{H}_2]] \tag{37}$$

where we have made the choice $r = I_n$.

Again, there is no way that this can vanish except for the single-component case. So these generalised KdV equations have no local Hamiltonian structures of third order in ∂.

On the other hand, the $\mathfrak{R}^{-1}J$ structure does have local many-component reductions. In place of (27) and (28) we have

$$\partial(q_{,t_3}) + q\partial^{-1}(rq_{,t_3}) + \partial^{-1}(q_{,t_3})rq = a\nabla_\eta \mathbb{H}_4' \tag{38}$$

$$r\partial^{-1}(q_{,t_3})r + r\partial^{-1}(q_{,t_3})r = -a\nabla_q \mathbb{H}_4. \tag{39}$$

Again, we must have r invertible and may write $qr = q'$ so that from (39) alone we obtain

$$q'_{,t_3} = -\tfrac{1}{2}a\partial\nabla_{q'}\mathbb{H}_4 \tag{40}$$

provided we impose no further constraints on the matrix q. These generalised KdV equations are studied in [8]. (Note that for r not square, say $n \times m$ with $n > m$, one may find an $m \times n$ matrix $\tilde{r}$ (not unique) such that $\tilde{r}r = I_m$ and define new dependent variables by $q' = qr$. Then (39) becomes

$$q'_{,t_3} = -\tfrac{1}{2}a\partial\nabla_{q'}\mathbb{H}_4 \tag{41}$$

and is locally Hamiltonian [3] in terms of the m^2 coordinates q'_{ij}.)

It should be emphasised that the local bi-Hamiltonian structure of the KdV does not generalise (within this class) and neither does the local MKdV structure.

The above discussion is germane to those equations associated, in the language of [2, 3], with the series of symmetric spaces AIII for $n = m$. The symmetric and antisymmetric reductions of (39), $q = q^{\mathrm{T}}$ and $q = -q^{\mathrm{T}}$ correspond to the symmetric spaces CI and DIII respectively. Their Hamiltonian structures are discussed in appendix 2. For class BDI symmetric spaces one no longer has $Q^+P^+ = Q^-P^- = 0$ for any Q and P, relations which were assumed in writing down (30) for instance. However, they contain as reductions members of the AIII series for which $n \neq m$ and are hence without local structures. (One would not expect a local structure to reduce to a non-local structure under a componentwise reduction.) Whilst somewhat negative, these results do clarify the relation between reductions and local Hamiltonian structures.

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Appendix 1

In this appendix we generalise an argument reproduced in [9]. From the general matrix problem (18)-(21) one obtains the matrix Riccati equation:

$$\gamma_{,x} + \zeta[A, \gamma] + Q - \gamma Q \gamma = 0 \tag{A1.1}$$

where γ is of the same form as Q. Define the two quantities

$$\mathfrak{K} = \mathrm{Tr}(Q\gamma) \tag{A1.2}$$

$$\mathfrak{H} = \mathrm{Tr}(AQ\gamma). \tag{A1.3}$$

Since γ has an expansion of the form

$$\gamma = \sum_{n=1}^{\infty} (a\zeta)^{-n} \gamma_n \tag{A1.4}$$

so do $\mathfrak{K}$ and $\mathfrak{H}$. In particular the coefficients of ζ^{-n} in $\mathfrak{H}$ are, up to constant factors, the integrands of the Hamiltonian functionals $\mathbb{H}_n$: $\mathbb{H}_n = C_n \int_{-\infty}^{\infty} \mathrm{d}x\, \mathfrak{H}_{n+1}$. We may write γ and $\mathfrak{H}$ as sums of parts corresponding to even and odd values of n:

$$\mathfrak{H} = \mathfrak{H}^{(+)} + \mathfrak{H}^{(-)} \qquad \gamma = \gamma^{(+)} + \gamma^{(-)}. \tag{A1.5}$$

From (A1.1) one easily derives an expression for $\mathfrak{K}$ as an x derivative:

$$\mathfrak{K} = \{\ln[\det(I - \gamma)]\}_{,x}. \tag{A1.6}$$

To consider the MKdV type reductions note that

$$\gamma^{\mathrm{T}}_{,x} - \zeta[A, \gamma^{\mathrm{T}}] + Q^{\mathrm{T}} - \gamma^{\mathrm{T}} Q^{\mathrm{T}} \gamma^{\mathrm{T}} = 0 \tag{A1.7}$$

which, together with (A1.1), implies

$$\gamma(Q, \zeta) = \gamma^{\mathrm{T}}(Q^{\mathrm{T}}, -\zeta) \tag{A1.8}$$

and consequently

$$\gamma^{(\pm)}(Q^{\mathrm{T}}) = \pm\gamma^{(\pm)\mathrm{T}}(Q). \tag{A1.9}$$

This provides us with an identity for the Hamiltonian densities associated with the Q and Q^{T} equations:

$$\mathfrak{H}^{(\pm)}(Q^{\mathrm{T}}) \pm \mathfrak{H}^{(\pm)}(Q) = \pm a \frac{n-m}{n+m} \mathfrak{K} \simeq 0 \tag{A1.10}$$

where $\simeq$ is the equivalence relation 'modulo x derivatives'. So under the reduction $Q = Q^{\mathrm{T}}$ we obtain $\mathfrak{H}^{(+)}(Q) \simeq 0$, i.e. $\mathfrak{H}_{2n} \simeq 0$ or $\mathbb{H}_{2n+1} = 0$, for $n > 0$. Further, if we write $Q = v + \eta$, where v is symmetric and η antisymmetric, then from (A1.10)

$$\mathfrak{H}^{(-)}(v + \eta) \simeq \mathfrak{H}^{(-)}(v - \eta) \tag{A1.11}$$

is an even function of η. Hence, for small η

$$\mathbb{H}_{2n}[Q] = \mathbb{H}_{2n}[v] + \mathrm{O}(\eta^2). \tag{A1.12}$$

For the KdV type reduction, $\boldsymbol{p} = \boldsymbol{r} =$ constant, write γ as

$$\gamma = \left(\begin{array}{c|c} & \boldsymbol{a} \\ \hline \boldsymbol{b} & \end{array}\right). \tag{A1.13}$$

Then

$$\mathfrak{H}^{(+)} = \mathrm{Tr}\{AQ\gamma^{(+)}\} \simeq -a\, \mathrm{Tr}\{\boldsymbol{r}\boldsymbol{a}^{(+)}\} \tag{A1.14}$$

by virtue of (A1.6), where the second trace is taken over $m \times m$ matrices. But (A1.1) gives us a Riccati equation for $\boldsymbol{a}$:

$$\boldsymbol{a}_{,x} + \zeta \boldsymbol{a}\boldsymbol{a} + \boldsymbol{q} - \boldsymbol{a}\boldsymbol{r}\boldsymbol{a} = 0 \tag{A1.15}$$

and from the part of this odd in ζ we deduce the relation

$$\operatorname{Tr}\{\boldsymbol{r}\boldsymbol{a}^{(+)}\} = \tfrac{1}{2}\{\ln[\det(I - (2/\zeta a)\boldsymbol{r}\boldsymbol{a}^{(-)})]\}_{,x}. \tag{A1.16}$$

Consequently $\mathfrak{H}^{(+)} \simeq 0$ and again $\mathbb{H}_{2n+1} = 0$.

Appendix 2

Equation (40) is written either as

$$a^2 \boldsymbol{q}_{,t_3} = (\boldsymbol{q}_{,xx} + 3\boldsymbol{q}\boldsymbol{r}\boldsymbol{q})_{,x} \tag{A2.1}$$

or as

$$a^2 \boldsymbol{q}'_{,t_3} = (\boldsymbol{q}'_{,xx} + 3\boldsymbol{q}'\boldsymbol{q}')_{,x}. \tag{A2.2}$$

Provided $\boldsymbol{q}$ suffers no constraints, the Hamiltonian structure (40) suffices.

However, when $\boldsymbol{q} = \boldsymbol{q}^{\mathrm{T}}$ and $\boldsymbol{r} = \boldsymbol{r}^{\mathrm{T}}$ (as is the case for the symmetric space CI) we write $\boldsymbol{q} = \boldsymbol{q}_0 + \boldsymbol{u} + \boldsymbol{u}^{\mathrm{T}}$ for $\boldsymbol{u}$ strictly upper triangular and $\boldsymbol{q}_0$ diagonal and then, as can be seen from the part of $\mathbb{H}_4$ quadratic in $\boldsymbol{q}$, proportional to $\operatorname{Tr}\{\boldsymbol{q}_{0,x}^2 + 2\boldsymbol{u}_{,x}^{\mathrm{T}}\boldsymbol{u}_{,x}\}$, we have

$$a^2 q_{ii,t_3} = -\frac{a}{2}\,\partial\left\{ r_{ik}^{-1} r_{ik}^{-1} \frac{\delta \mathbb{H}_4}{\delta q_{kk}} \right\} \tag{A2.3}$$

where we sum over k only and, for $j > i$,

$$a^2 q_{ij,t_3} = -\frac{a}{4}\,\partial\left(r_{ik}^{-1} r_{lj}^{-1} \frac{\delta \mathbb{H}_4}{\delta q_{kl}} \right) \tag{A2.4}$$

where we sum over k and l with $l > k$.

Again, when $\boldsymbol{q}$ and $\boldsymbol{r}$ are antisymmetric (the case DIII), we write $\boldsymbol{q} = \boldsymbol{u} - \boldsymbol{u}^{\mathrm{T}}$ for $\boldsymbol{u}$ strictly upper triangular, $\mathbb{H}_4$ is proportional to $\operatorname{Tr}\{2\boldsymbol{u}_{,x}^{\mathrm{T}}\boldsymbol{u}_{,x}\}$ and equation (A2.4) only applies.

Note that the above reductions are equivalent to reductions of (A2.2) of the form $\boldsymbol{q}'^{\mathrm{T}}\boldsymbol{r} = \pm \boldsymbol{r}\boldsymbol{q}'$ with $\boldsymbol{r} = \boldsymbol{r}^{\mathrm{T}}$ or $\boldsymbol{r} = -\boldsymbol{r}^{\mathrm{T}}$.

More generally, we might consider (A2.1) under the reduction $\boldsymbol{r}'\boldsymbol{q} = \boldsymbol{q}^{\mathrm{T}}\boldsymbol{r}'$ with $\boldsymbol{r} = \boldsymbol{r}'^{\mathrm{T}}$ which further implies $\boldsymbol{r}'\boldsymbol{r} = \boldsymbol{r}^{\mathrm{T}}\boldsymbol{r}'$. But this is equivalent to the CI case. Likewise the reduction with $\boldsymbol{r}'\boldsymbol{q} = -\boldsymbol{q}^{\mathrm{T}}\boldsymbol{r}'$ reduces to DIII.

Such reductions are not without interest. They include, for instance, the almost persymmetric reduction, $q_{i,j} = q_{n+1-j,n+1-i}$. In the case $n = 2$ this gives us the three-component system:

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}_{,t_3} = \partial \begin{pmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta/\delta p_1 \\ \delta/\delta p_2 \\ \delta/\delta p_3 \end{pmatrix} \int \mathrm{d}x\, \mathbb{H} \tag{A2.5}$$

where $\mathbb{H} = -p_{2,x}^2 - p_{1,x}p_{3,x} - 2p_2^3 - 6p_1p_2p_3$ and this is not equivalent to the symmetric reduction.

It is not clear whether such reductions and their composites exhaust all possible reductions.

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